DEEPLY PENETRATING TRANSVERSE WAVES IN A ROTATING VISCOUS FLUID

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It is shown that as the frequency of rotation of a fluid approaches the half frequency of the axial oscillation of a disk immersed in it there is an abrupt increase in the depth of penetration of viscous waves which are excited by the oscillations of the disk.

1. Formulation of the problem. In the paper [1] the axial oscillations of a disk in a rotating viscous incompressible fluid have been investigated. The case in which the angular velocity of rotation ω_0 approaches half of the frequency of oscillation Ω was excluded from consideration here, since certain conditions necessary in the derivation of the boundary correction formulas carried out in [1] would be violated.

It is exactly this case $2\omega_0 \approx \Omega$ that is the subject of the present paper; however, the question of the boundary corrections is avoided.

Let a viscous incompressible fluid infinite in the radial direction rotate about the Oz axis with a constant angular velocity ω_0, and let a disk of infinite radius immersed in it rotate in the fluid while simultaneously performing small axial oscillations about the axis of rotation with amplitude $\varphi_0 \ll 1$, frequency Ω and damping coefficient γ. The fluid above and below is assumed to be unbounded or bounded by plane surfaces parallel to the surface of the disk.

The angular velocity $\varphi^{\cdot}$ of the disk depends on the time t in the following way:

$$\varphi^{\cdot} = \omega_0 + i\alpha\varphi_0 e^{i\alpha t} \qquad (\alpha = \Omega + i\gamma) \tag{1.1}$$

Here α is the complex oscillation frequency which is assumed to be quasi-harmonic $(\gamma \ll \Omega)$.

Taking the axial symmetry of the problem into consideration, we shall represent the velocity and pressure distributions in the fluid in cylindrical coordinates r, φ, z in the form of a sum of the "rotary" and "oscillatory" terms [1]

$$v_r = rw_r(z)\, e^{i\alpha t}, \qquad v_\varphi = \omega_0 r + rw_\varphi(z)\, e^{i\alpha t}, \qquad v_z = w_z(z)\, e^{i\alpha t} \tag{1.2}$$

$$p = p_0 + {}^1/_2\rho\omega_0^2 r^2 + p_1(r, z)\, e^{i\alpha t} \qquad (p_0 = \text{const}) \tag{1.3}$$

ρ is the density of the fluid.

Assuming the amplitude of the oscillations to be so small that $\Omega\varphi_0 \ll \omega_0$, the Navier-Stokes equations can be linearized with respect to the quantities w_r, w_φ, w_z and, supplementing it with the continuity equation, the following system of linearized equations can be obtained for the four unknown functions w_r, w_φ, w_z and p_1:

$$ia w_r - 2\omega_0 w_\varphi = -\frac{1}{\rho}\frac{1}{r}\frac{dp_1}{dr} + \nu\frac{d^2 w_r}{dz^2}, \qquad 2w_r + \frac{dw_z}{dz} = 0 \tag{1.4}$$

$$ia w_\varphi + 2\omega_0 w_r = \nu\frac{d^2 w_\varphi}{dz^2}, \qquad ia w_z = -\frac{1}{\rho}\frac{dp_1}{dz} + \nu\frac{d^2 w_z}{dz^2} \tag{1.5}$$

Here $\nu = \eta/\rho$ is the kinetic viscosity of the fluid and η is the dynamic viscosity.

2. Solution of the system. The solution of the system of equations (1.4) and (1.5) has the form

$$2w_r(z) = B^{(+)}\exp(ik^{(+)}z) + C^{(+)}\exp(-ik^{(+)}z) + B^{(-)}\exp(ik^{(-)}z) + C^{(-)}\exp(-ik^{(-)}z) - \frac{2i\alpha A}{\alpha^2 - 4\omega_0^2} \tag{2.1}$$

$$2iw_\varphi(z) = B^{(+)}\exp(ik^{(+)}z) + C^{(+)}\exp(-ik^{(+)}z) - B^{(-)}\exp(ik^{(-)}z) - C^{(-)}\exp(-ik^{(-)}z) + \frac{4i\omega_0 A}{\alpha^2 - 4\omega_0^2} \tag{2.2}$$

$$w_z(z) = \frac{i}{k^{(+)}}\left(B^{(+)}\exp(ik^{(+)}z) - C^{(+)}\exp(-ik^{(+)}z)\right) + \frac{i}{k^{(-)}}\left(B^{(-)}\exp(ik^{(-)}z) - C^{(-)}\exp(-ik^{(-)}z)\right) + \frac{2i\alpha A}{\alpha^2 - 4\omega_0^2}z + D \tag{2.3}$$

$$-\frac{1}{\rho}p_1(r, z) = \frac{1}{2}Ar^2 - \nu\frac{dw_z(z)}{dz} + i\alpha\int_a^z w_z(u)\,du \tag{2.4}$$

The solution contains seven coefficients: A, B^+, B^-, C^+, C^- and D and the constant of integration in (2.4).

3. Boundary conditions. In view of the lack of direct communication between the fluid located above the disk and the fluid below it, they can be considered separately. Then, to determine the coefficients of the solution (2.1) to (2.4) we have three boundary conditions of the surface of the disk ($z = 0$) associated with the law of its motion (1.1) and with Formulas (1.2)

$$w_r(0)=0, \quad w_\varphi(0)=i\alpha\varphi_0, \quad w_z(0)=0 \tag{3.1}$$

Three more boundary conditions are given on the sutface which bounds the fluid above or below (we shall designate it as $z = H$). If this surface is rigid and moves just like the basic disk, the conditions for $z = H$ then coincide with (3.1). If it is a rigid and nonoscillating plane, then

$$w_r(H) = 0, \quad w_\varphi(H) = 0, \quad w_z(H) = 0 \tag{3.2}$$

In the cases (3.1) and (3.2) the boundary conditions do not affect the function p_1, and therefore the constant of integration in (2.4) is not determined by them.

But if the upper boundary of the fluid is a free surface, the boundary conditions on it are then determined in the usual way (cf. [2] page 69) with the help of the momentum flux tensor. It is easy to be convinced that they reduce to the identities

$$w_r'(H) = 0, \qquad w_\varphi'(H) = 0, \qquad p_1(r, H) = 0 \tag{3.3}$$

Although there are three equalities here, as in the case of (3.1) and (3.2) also, one of them breaks up into two, as will be shown in Section 5. Thus, the conditions (3.1) and (3.3) together determine all seven coefficients of the solution (2.1) to (2.4).

4. Wave number analysis. Formulas (2.1) and (2.2) show that oscillations of a disk in a viscous fluid generate two waves with mutually opposed circular polarizations and with wave numbers $k^{(+)}$ (plus-wave) and $k^{(-)}$ (minus-wave). The wave numbers $k^{(\pm)}$ are defined by Formula

$$k^{(\pm)2} = -i\frac{\alpha \pm 2\omega_0}{\nu} \tag{4.1}$$

In the following we shall for definiteness consider that

$$\operatorname{Im} k^{(\pm)} > 0 \tag{4.2}$$

Formula (4.1) differes from the corresponding Formula (2.1) of [1] only in that in the latter the frequency Ω appears in place of $\alpha = \Omega + i\gamma$. In view of the condition $\gamma \ll \Omega$ such an inaccuracy (deliberately assumed in [1] does not influence the validity of the results, since the case $2\omega_0 \approx \Omega$ is excluded from consideration. However, if $2\omega_0 \approx \Omega$, the case

$$|\Omega - 2\omega_0| \ll \gamma \tag{4.3}$$

for which the difference between Formula (4.1) and Formula (2.1) of [1] becomes very important, is possible.

We shall introduce the notation

$$k^{(\pm)} = \sigma^{(\pm)} + i\tau^{(\pm)} \tag{4.4}$$

and with the help of Formula (4.1) determine the real and imaginary parts of the wave numbers $k^{(\pm)}$:

$$\tau^{(\pm)} = \frac{1}{\sqrt{2\nu}}\left(\sqrt{\gamma^2 + (\Omega \pm 2\omega_0)^2} - \gamma\right)^{1/2} \tag{4.5}$$

$$\sigma^{(\pm)} = -\frac{1}{\sqrt{2\nu}}\frac{\Omega \pm 2\omega_0}{\left(\sqrt{\gamma^2 + (\Omega \pm 2\omega_0)^2} - \gamma\right)^{1/2}} \tag{4.6}$$

In view of the condition $\gamma \ll \Omega$ we always have $\Omega + 2\omega_0 \gg \gamma$. Therefore Formulas

$$\tau^{(+)} = -\sigma^{(+)} = \left(\frac{\Omega + 2\omega_0}{\nu}\right)^{1/2} \quad (\text{for } \Omega + 2\omega_0 \gg \gamma) \tag{4.7}$$

are applicable for $\tau^{(+)}$ and $\sigma^{(+)}$ with a high degree of accuracy.

Hence it follows that the plus-wave penetration depth $\lambda^{(+)}$ and its length are related by

$$L^{(+)} = 2\pi\lambda^{(+)} = 2\pi\left(\frac{\nu}{\Omega + 2\omega_0}\right)^{1/2} \qquad \left(\lambda^{(+)} = \frac{1}{\tau^{(+)}}\ L^{(+)} = \frac{2\pi}{|\sigma^{(+)}|}\right) \tag{4.8}$$

Under such conditions there is realized in practice not a wave but an oscillation of a layer adjacent to the disk with thickness of order $\frac{1}{2}\lambda^{(+)}$ ([2], page 112). Moreover, if it is taken into consideration that the quantity $\lambda^{(+)}$ is very small for a fluid with a not very high viscosity and for frequencies Ω convenient for carrying out measurements, then observation of the results of reaching by this wave a surface at a distance H from the disk and being reflected from this surface is unrealizable.

The minus-wave has the same character as long as $|\Omega - 2\omega_0| \gg \gamma$. Then

$$L^{(-)} = 2\pi\lambda^{(-)} = 2\pi\left(\frac{\nu}{|\Omega - 2\omega_0|}\right)^{1/2} \qquad (\text{for } |\Omega - 2\omega_0| \gg \gamma) \tag{4.9}$$

However, when $2\omega_0 \to \Omega$, it follows from Formulas (4.5) and (4.6) that

$$\lim_{2\omega_0 \to \Omega} \tau^{(-)} = 0, \qquad \lim_{2\omega_0 \to \Omega \pm 0} \sigma^{(-)} = \pm\left(\frac{\gamma}{\nu}\right)^{1/2} \tag{4.10}$$

Hence

$$\lim_{2\omega_0 \to \Omega} \lambda^{(-)} = \infty, \qquad \lim_{2\omega_0 \to \Omega} L^{(-)} = 2\pi\left(\frac{\nu}{\gamma}\right)^{1/2} \tag{4.11}$$

Consequently, provided that $\gamma \neq 0$, the depth of penetration of a minus-wave in the region of frequencies of rotation close to half of the frequency of the oscillations can be comparable to its length and can even considerably exceed it. In the presence of such deeply penetrating waves resonance phenomena associated with the formation of a standing minus-wave in the space between the disk (the wave generator) and the reflecting surface should be easily observed. These effects should be observed as long as the distance H does not exceed the depth of penetration $\lambda^{(-)}$, defined in accordance with (4.5) by Formula

$$\lambda^{(-)} = \frac{\sqrt{2\nu}}{(\sqrt{\gamma^2 + (\Omega - 2\omega_0)^2} - \gamma)^{1/2}} \tag{4.12}$$

The distance H_0 at which the resonance effects damp out can serve as an experimental estimate of the quantity $\lambda^{(-)}$.

Standing waves are generated between the oscillating surfaces for $H = \frac{1}{2}nL^{(-)}$, between the disk and the nonoscillating surface for $H = \frac{1}{4}(2n-1)L^{(-)}$ and between the disk and the free surface for $H = \frac{1}{2}nL^{(-)}$, where $n = 1, 2, \ldots$ and $L^{(-)}$ is determined with the help of Formula (4.6)

$$L^{(-)} = \frac{2\pi\sqrt{2\nu}\,(\sqrt{\gamma^2 + (\Omega \pm 2\omega_0)^2} - \gamma)^{1/2}}{|\Omega - 2\omega_0|} \tag{4.13}$$

Therefore, the phenomena dependent on the propagation conditions of the minus-wave must show a periodic dependence on H, whose study allows the quantity $L^{(-)}$ to be measured.

5. Oscillations of a disk under a free surface. A freely suspended and oscillating disk not only generates oscillations in a viscous fluid but also itself feels their effect. Therefore, it itself can serve as an indicator of the resonance effects predicted in the previous Section. These effects are reflected in the periodical dependency of the oscillation character of the disk on the distance H.

The frequency and damping of the oscillations of the disk can be found by the method described in [1], knowing the solution of the system (1.4) and (1.5). We shall investigate the dependence on H of only the second of these quantities. For this we shall write down the moment of force M acting on the surface of the disk (cf. (3.1) in [1]) and the contribution of this moment to the damping $\Delta\gamma$ (the quantity γ is additive) (cf. (3.6) in [1]):

$$M = {}^1/_2\,\pi\eta R^4 w_\varphi{}'(0)\, e^{i\alpha t}, \qquad \Delta\gamma = -\frac{\operatorname{Im}(Me^{-i\alpha t})}{2I\Omega\varphi_0} \tag{5.1}$$

Here R and I are respectively the radius of the disk and the moment of inertia of the oscillating system. Up to this point the question has concerned oscillations of a disk of infinite radius, therefore the use of Formulas (5.1) is associated with the neglect of boundary effects.

Only the case of the free surface for which the expression for M turns out to be comparatively simple is considered here (the free surface can be regarded as a plane if the condition for smallness of the curvature of the meniscous of the rotating fluid $\omega_0{}^2R/g \ll 1$ is observed, where g is the acceleration of free fall). Moreover, the interaction of the upper surface of the disk only with the free surface is investigated, it is assumed that the distance between the disk and the bottom of the container does not vary, and the contribution to the damping of the oscillation does not depend on H. According to (2.4), we have the last of the conditions (3.3) in the form

$$\frac{1}{2}Ar^2 - \nu w_z{}'(H) + i\alpha\int_a^H w_z(z)\,dz = 0 \tag{5.2}$$

In view of the independence of w_z from r there is obtained $A = 0$. Then the boundary conditions (3.1) and (3.3) turn out to be sufficient to determine the other six coefficients of the solution (2.1) to (2.4), of which we shall write down only $B^{(\pm)}$ and $C^{(\pm)}$:

$$B^{(\pm)} = \mp \alpha\varphi_0 \frac{e^{-ik^{(\pm)}H}}{e^{ik^{(\pm)}H} + e^{-ik^{(\pm)}H}}, \qquad C^{(\pm)} = \mp \alpha\varphi_0 \frac{e^{ik^{(\pm)}H}}{e^{ik^{(\pm)}H} + e^{-ik^{(\pm)}H}} \tag{5.3}$$

since the other coefficients are not necessary to calculate the quantity M which, according to (5.1), (2.2) and (5.3) is

$$M = {}^1/_4\, \pi R^4 \eta i\alpha \left(k^{(+)} \tan k^{(+)} H + k^{(-)} \tan k^{(-)} H\right) \tag{5.4}$$

Substituting (5.4) into (5.2) and taking into account the inequality $\gamma \ll \Omega$ gives

$$\Delta\gamma = \frac{\pi R^4 \eta}{8I} \left(\frac{\tau^{(+)} \sinh 2\tau^{(+)} H - \sigma^{(+)} \sin 2\sigma^{(+)} H}{\cos 2\sigma^{(+)} H + \cosh 2\tau^{(+)} H} + \frac{\tau^{(-)} \sinh 2\tau^{(-)} H - \sigma^{(-)} \sin 2\sigma^{(-)} H}{\cos 2\sigma^{(-)} H + \cosh 2\tau^{(-)} H} \right) \tag{5.5}$$

Numerical calculations with this formula indicate that for increasing but still sufficiently small H ($H \gg \lambda^{(+)}$) the first term in parentheses ceases to depend on it. If $2\omega_0 \approx \Omega$, then the second term depends periodically on H (the longer the closer ω_0 to $\Omega/2$) and this dependency gradually dies out, coming to naught for $H \gg \lambda^{(-)}$; for such values of H the damping of the oscillation has the value computed in [1] and is typical for oscillations of a disk in an unbounded fluid.

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